WHEN $\mathrm{Min}(G)^{-1}$ HAS A CLOPEN π-BASE

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ABSTRACT. It is our aim to add to the flourishing collection of knowledge centered on the space of minimal prime subgroups of a given lattice-ordered group. Specifically, we are interested in the inverse topology. In general, this space is compact and T_1, but need not be Hausdorff. In [18] W. Wm. McGovern: *Neat rings*, Journal of Pure and Applied Algebra, **205** (2006), 243-265, it was shown that this space is a boolean space (i.e. a compact zero-dimensional and Hausdorff space) if and only if the ℓ-group in question is weakly complemented. A slightly weaker topological property than having a base of clopen subsets is having a clopen π-base. Recall that a π-base is a collection of non-empty open subsets such that every non-empty open subset of the space contains a member of the π-base; obviously, a base is a π-base. In what follows we classify when the inverse topology on the space of prime subgroups has a clopen π-base.

1. Introduction

Throughout $(G, +, 0, \vee, \wedge)$ will denote a lattice-ordered group. Unless otherwise noted, we do not assume that G is abelian. Recall that an ℓ-subgroup H of G is **convex** if whenever $0 \leq g \leq h$ for some $h \in H$, then $g \in H$. The set of convex ℓ-subgroups of G is denoted by $\mathcal{C}(G)$. The intersection of any collection of convex ℓ-subgroups is itself a convex ℓ-group and therefore $\mathcal{C}(G)$ is a complete lattice when partially ordered by inclusion. We shall denote the convex ℓ-subgroup generated by $g \in G$, by $\mathfrak{G}(g)$ and call this the **principal convex ℓ-subgroup generated** by g.

A (proper) convex ℓ-subgroup P of G is said to be a **prime** subgroup if whenever $a \wedge b = 0$, then either $a \in P$ or $b \in P$. The collection of all prime subgroups is known as the prime spectrum of G and is denoted by $\mathrm{Spec}(G)$. By Zorn's Lemma, given any $0 < a \in G$ there is a convex ℓ-subgroup that is maximal with respect to not containing a. Such a subgroup is called a **value** of a and we use $\mathrm{Val}(a)$ to denote the set of values of a. It is known that values are prime subgroups, and not conversely. In particular, $\mathrm{Spec}(G)$ is non-empty when G is non-trivial. Lattice-ordered groups have the feature that the collection of primes containing a given prime forms a chain, i.e. $\mathrm{Spec}(G)$ is a root system. Since the intersection of a chain of prime subgroups is again a prime subgroup, it follows that minimal prime subgroups exist; the collection of these is denoted by $\mathrm{Min}(G)$. It is this set that captivates our interest.

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The set $\mathrm{Min}(G)$ can be equipped with two topologies. Formally, the **hull-kernel topology** on $\mathrm{Min}(G)$ has as a base of open sets the collection

$$\mathcal{H} = \{U(g) : g \in G\}$$

where $U(g) = \{P \in \mathrm{Min}(G) : g \notin P\}$. The collection $\mathcal{H}$ is closed under finite unions and finite intersections. The complement of $U(g)$ is denoted by $V(g) = \mathrm{Min}(G) \smallsetminus U(g)$ and we let

$$\mathcal{I} = \{V(g) : g \in G\}.$$

The set $\mathcal{I}$ is obviously also closed under finite unions and finite intersections. The **inverse topology** on $\mathrm{Min}(G)$ is the topology generated by the collection $\mathcal{I}$. Topologically speaking, we distinguish between the topologies by letting $\mathrm{Min}(G)$ denote the space equipped with the hull-kernel topology, and letting $\mathrm{Min}(G)^{-1}$ denote the space equipped with the inverse topology.

Lemma 1.1. *Let G be an ℓ-group and $a, b, g \in G^+$. Then*
(a) $U(a) \cap U(b) = U(a \wedge b)$, *and*
(b) $V(a) \cap V(b) = V(a \vee b)$.
(c) $V(g) = \mathrm{Min}(G)$ *if and only if* $g = 0$.
(d) $V(g) = \emptyset$ *if and only if g is a weak order unit.*

Recall that an element $0 \leq g \in G$ is called a **weak order unit** of G, whenever it satisfies the property that for all $h \in G$, $g \wedge h = 0$ implies $h = 0$.

A space is said to be **zero-dimensional** if it has a base of clopen subsets. The space $\mathrm{Min}(G)$ is a zero-dimensional Hausdorff space; each member of $\mathcal{H}$ is a clopen subset. However, the hull-kernel topology on $\mathrm{Min}(G)$ is not always compact. On the other hand, the inverse topology on $\mathrm{Min}(G)^{-1}$ is always compact and T_1, but not always zero-dimensional.

The ℓ-group G is called **complemented** when it has the property that for each $0 \leq g \in G$ there is an $0 \leq h \in G$ such that $g \wedge h = 0$ and $g \vee h$ is a weak order unit of G. (An element $g \in G^+$ for which there is such an $h \in G^+$ is called **complemented** and such a pair $g, h \in G^+$ is called a **complementary pair**.) Theorem 2.2 of [7] states and proves that G is complemented if and only if $\mathrm{Min}(G)$ is a compact Hausdorff space. Later, it was pointed out that G is complemented if and only if $\mathrm{Min}(G) = \mathrm{Min}(G)^{-1}$. (This last equivalence was first proved for abelian groups in [18] and later mentioned in [14] that the proof carries over for all ℓ-groups *mutatis mutandis*. Much of the above can be phrased in terms of lattice theory; foundational results in the theory can be traced back to the work of Kist [13] and Speed [19].) Formally, we state this.

Theorem 1.2. *For an ℓ-group G the following are equivalent.*
(1) *G is complemented.*
(2) $\mathrm{Min}(G)$ *is compact.*
(3) $\mathrm{Min}(G) = \mathrm{Min}(G)^{-1}$.

It follows that if G is complemented, then $\mathrm{Min}(G)^{-1}$ is a **boolean space** (that is, compact, Hausdorff, and zero-dimensional). In [18] and [14] the authors classify when $\mathrm{Min}(G)^{-1}$ is a boolean space. The important algebraic notion is that of a

weakly complemented ℓ-group: whenever $g \wedge h = 0$ there is a complementary pair $x, y \in G^+$ such that $g \leq x$ and $h \leq y$.

Theorem 1.3. *Let G be an ℓ-group. Then G is weakly complemented if and only if $\mathrm{Min}(G)^{-1}$ is a boolean space.*

In the paper [2], the authors generalized the notion of a weakly complemented ℓ-group with the goal of classifying when $\mathrm{Min}(G)^{-1}$ is a Hausdorff space. The ℓ-group G is called **lamron** if whenever $g, h \in G^+$ such that $g \wedge h = 0$, then there are $x, y \in G^+$ such that $g \leq x$, $h \leq y$, $g \wedge y = 0 = h \wedge x$, and $x \vee y$ is a weak order unit. (Notice that in this definition the elements x and y need not be a complementary pair.)

In [3], the authors investigated the space of maximal d-subgroups of G, denoted $\mathrm{Max}_d(G)$, and made a connection between a lamron ℓ-group G and the space $\mathrm{Max}_d(G)$. We deviate and recall the fundamentals of d-subgroups.

First, for any set $S \subseteq G$, the **polar of** S is the set

$$S^\perp = \{g \in G : \forall s \in S, |g| \wedge |s| = 0\}.$$

Fact, every polar is a convex ℓ-subgroup. When $S = \{f\}$ we instead write $f^\perp$ and call this the polar of f. Notice that using this notation we could have defined $g \in G$ to be a weak order unit if $g^\perp = \{0\}$. The convex ℓ-subgroup $f^{\perp\perp}$ is called the **principal polar of** f and such objects are used to define d-subgroups. An $H \in \mathcal{C}(G)$ is called a d-**subgroup** if for all $h \in H$, $h^{\perp\perp} \subseteq H$. Notice that a proper d-subgroup cannot contain any weak order units. When G has a weak order unit, then maximal d-subgroups exist and are indeed prime subgroups. In the sequel the only topology considered on $\mathrm{Max}_d(G)$ is the hull-kernel. For each $g \in G$, let $U_d(g) = \{M \in \mathrm{Max}_d(G) : g \notin M\}$.

Remark 1.4. The interested or unfamiliar reader should check the articles [11] and [12] for more information on d-subgroups in the context of archimedean vector lattices. According to M. Darnell [8], d-subgroups were originally called z-subgroups by Bigard. However, given that z-ideals are a different well-studied concept in the theory of continuous functions, nowadays it appears that the nomenclature of d-subgroups is appropriate. Azarpanah and others [1], within the confines of ring theory, call these z°-ideals. In our experience, we have found that searching for d-ideals is easier than for z°-ideals.

Lemma 1.5. *Let G be an ℓ-group and $a, b, g \in G^+$. Then*
(a) *$U_d(a) \cap U_d(b) = U(a \wedge b)$, and*
(b) *$U_d(a) \cup U_d(b) = U_d(a \vee b)$.*
(c) *$U_d(g) = \mathrm{Max}_d(G)$ if and only if g is a weak order unit.*

Example 1.6. In general, it is not the case that $\cap \mathrm{Max}_d(G) = \{0\}$. For example, let G be the lexicographic extension of $\mathbb{Z}$ over $H = \oplus_{n \in \mathbb{N}} \mathbb{Z}$, the direct sum of countably many copies of $\mathbb{Z}$. Then $\mathrm{Max}_d(G) = \{H\} \neq \{0\}$. It is true that if G is archimedean and has a weak order unit, then $\cap \mathrm{Max}_d(G) = \{0\}$; more on archimedean ℓ-groups later. Also, in §4, we characterize those elements satisfying $V_d(g) = \mathrm{Max}_d(G)$.

We end this section with some interesting observations from [4]. We assume in all that follows that G possesses a weak order unit.

(i) Each maximal d-subgroup is a prime subgroup. The set $\mathrm{Max}_d(G)$ can be equipped with the hull-kernel topology making it into a compact Hausdorff space.
(ii) For each $P \in \mathrm{Min}(G)$ there is a unique $\mathfrak{d}(P) \in \mathrm{Max}_d(G)$ containing it, giving rise to a continuous surjective map $\mathfrak{d} : \mathrm{Min}(G)^{-1} \longrightarrow \mathrm{Max}_d(G)$.
(iii) The map $\mathfrak{d}$ is a bijection (and hence a homeomorphism) if and only if G is a lamron ℓ-group.
(iv) $\mathrm{Min}(G) = \mathrm{Max}_d(G)$ if and only if G is complemented.

The following result is instrumental in our thinking.

Proposition 1.7. *Let G be a lamron ℓ-group. The following are equivalent.*
(1) *G is a weakly complemented ℓ-group.*
(2) *$\mathrm{Max}_d(G)$ is zero-dimensional.*
(3) *$\mathrm{Min}(G)^{-1}$ is zero-dimensional.*

The main question that intrigues us is whether it is possible to have an ℓ-group G for which $\mathrm{Max}_d(G)$ is zero-dimensional while $\mathrm{Min}(G)^{-1}$ is not. Obviously, such an ℓ-group must not be lamron. It is within this framework that we were led to the current work.

2. Clopen π-Bases

We start this section with the central topological definition pertaining to this article.

Definition 2.1. Let X be a topological space. A collection $\mathcal{B}$ of (non-empty) open subsets of X is called a *π-base* if every non-empty open subset of X contains some member of $\mathcal{B}$. Obviously, a base of (non-empty) open sets is a π-base.

A clopen π-base is a π-base for which each element in it is a clopen subset. For example, any space with a dense set of isolated points has a clopen π-base. The connection here is that the study of clopen subsets of $\mathrm{Min}(G)^{-1}$ is akin to the study of complemented elements of G.

Lemma 2.2. [18, Lemma 5.1] *A subset $K \subseteq \mathrm{Min}(G)^{-1}$ is clopen if and only if $K = U(e)$ for some complemented $e \in G^+$. Furthermore, if $U(e)$ is a clopen subset of $\mathrm{Min}(G)^{-1}$, then e is a complemented element.*

Thus, the question of when $\mathrm{Min}(G)^{-1}$ has a clopen π-base can be answered efficiently as follows. By a **proper complemented** element we mean a complemented element which is not a weak order unit. Equivalently, a proper complemented element is an $e \in G^+$ such that $U(e)$ is a proper subset of $\mathrm{Min}(G)^{-1}$.

Theorem 2.3. *The space $\mathrm{Min}(G)^{-1}$ has a clopen π-base if and only if for every non-weak order unit $0 < g \in G$ there is a proper complemented element $e \in G^+$ such that $g \leq e$.*

Proof. Suppose $\mathrm{Min}(G)^{-1}$ has a clopen π-base, say $\mathcal{B}$. By Lemma 2.2, each member of $\mathcal{B}$ is of the form $V(f)$ for some complemented element $f \in G^+$. We assume that each $V(f) \neq \emptyset$, hence each f is not a weak order unit. Take $0 < g \in G^+$ and suppose that g is not a weak order unit, then $V(g) \neq \emptyset$. So, there exists a $V(f) \in \mathcal{B}$ such that $V(f) \subseteq V(g)$. Since $V(f) = V(f) \cap V(g) = V(f \vee g)$, it follows that $f \vee g$ is a proper complemented element.

Conversely, suppose that each positive non-weak order unit of G is surpassed by a proper complemented element. Take a basic open set of $\mathrm{Min}(G)^{-1}$, say $V(g)$ with $0 < g$, and without loss of generality we assume that g is not a weak order unit. By hypothesis, there exists a proper complemented element $g \leq e$. Then $V(e) \subseteq V(g)$. Since $\emptyset \neq V(e)$ it follows that the collection

$$\mathcal{B} = \{V(e) : 0 < e \text{ is a proper complemented element of } G\}$$

is a clopen π-base of $\mathrm{Min}(G)^{-1}$. □

This has led us to the following new type of subgroup.

Definition 2.4. Let $H \in \mathcal{C}(G)$. We call H a *c-subgroup* of G if for all $0 < h \in H$ there exists a complemented element $e \in H^+$ such that $h \leq e$. We let $\mathcal{C}_c(G)$ denote the collection of c-subgroups of G.

Some simple observations are in order. For c-subgroups to exist it is necessary that G possess a weak order unit, in which case, by convention and definition, the trivial subgroups are both c-subgroups. Thus, any convex ℓ-subgroup of G containing a weak order unit is a c-subgroup. Moreover, a convex ℓ-subgroup is a c-subgroup if and only if it is generated (*vis-a-vis* convexity) by its complemented elements. We find it useful to notate the set of positive complemented elements.

Definition 2.5. Denote the set of complemented elements of G by $c(G)$;

$$c(G) = \{g \in G^+ : g \text{ is a complemented element of } G\}.$$

Lemma 2.6. *The collection $c(G)$ is a sublattice of G^+.*

Proof. Let $g_1, g_2 \in c(G)$ and choose $h_1, h_2 \in G^+$ so that $g_i \wedge h_i = 0$ and $g_i \vee h_i$ is a weak order unit ($i = 1, 2$). We prove that $g_1 \vee g_2$ is complemented with complement $h = h_1 \wedge h_2$. First,

$$\begin{aligned}(g_1 \vee g_2) \wedge h &= (g_1 \wedge h) \vee (g_2 \wedge h)\\ &= (g_1 \wedge h_1 \wedge h_2) \vee (g_2 \wedge h_1 \wedge h_2)\\ &= 0 \vee 0\\ &= 0.\end{aligned}$$

Second, to show that $(g_1 \vee g_2) \vee h$ is a weak order unit, let $t \in G$ satisfy

$$t \wedge [(g_1 \vee g_2) \vee h] = 0.$$

Then $[t \wedge (g_1 \vee g_2)] \vee (t \wedge h) = 0$, whence both $t \wedge (g_1 \vee g_2) = 0$ and $t \wedge (h_1 \vee h_2) = 0$. The former implies that both $t \wedge g_1 = 0$ and $t \wedge g_2 = 0$, whence the element $t \wedge h_2$ has the property that it is disjoint from both g_1 and h_1. This implies it is disjoint from $g_1 \vee h_1$, a weak order unit. Consequently, $t \wedge h_2 = 0$. But then t is disjoint from $g_2 \vee h_2$, a weak order unit. Therefore, $t = 0$. □

It is clear that $c(G)$ contains each weak order unit. It is entirely possible that 0 is the only proper complemented element. Example 7.1 is such a case. In the sequel we will be interested in those c-subgroups which do not contain any weak order unit, calling these **proper c-subgroups**. It follows that the only proper c-subgroup in Example 7.1 is $\{0\}$.

3. The frame of c-subgroups

As was already pointed out, $\mathcal{C}(G)$ is a complete lattice under inclusion. It is also a distributive lattice and furthermore, finite meets distribute over arbitrary joins. All of this together is easily stated by saying that $\mathcal{C}(G)$ is a frame. It is known that the collection of d-subgroups of G, denoted $\mathcal{C}_d(G)$, is also a frame ([16, §5]). In general, $\mathcal{C}_d(G)$ is not a subframe of $\mathcal{C}(G)$. We consider $\mathcal{C}_c(G)$. For the next result recall that the join of a collection of convex ℓ-subgroups is the subgroup generated by the collection ([8]).

Proposition 3.1. *Let $A, B \in \mathcal{C}_c(G)$ and $\{H_i\} \subseteq \mathcal{C}_c(G)$. Let $g \in G^+$. Then*
(a) *$A \cap B \in \mathcal{C}_c(G)$,*
(b) *$\bigvee H_i \in \mathcal{C}_c(G)$, and*
(c) *If $g \in c(G)$, then $g^\perp \in \mathcal{C}_c(G)$.*

Proof. (a) If $A \cap B = \{0\}$, then we are done. Otherwise, choose $0 < h \in A \cap B$. By hypothesis, there are complemented elements, say $a \in A^+$ and $b \in B^+$ such that $h \le a$ and $h \le b$. Therefore, $h \le a \wedge b$, with the latter a complemented element belonging to $A \cap B$.

(b) It suffices to show that $H = \bigvee H_i$ is a c-subgroup. To that end, let $h \in H^+$. Then there is a collection $h_1, \dots, h_n \in G$ such that $h_i \in H_{j_i}$ and $h \le h_1 \vee \dots \vee h_n$. Since each H_{j_i} is a c-subgroup, there is a complemented $e_i \in H_{j_i}$ such that $h_i \le e_i$ and therefore,

$$h \le e_1 \vee \dots \vee e_n$$

with the latter a complemented element belonging to $H = \bigvee H_i$.

(c) Let h be a complement of g. Then for any $0 \le t \in g^\perp$, $t \vee h \in g^\perp$ is also a complement of g as $(t \vee h) \vee g$ is a weak order unit and $g \wedge (t \vee h) = 0$. □

Theorem 3.2. *Let G be an ℓ-group. Then $\mathcal{C}_c(G)$ is a subframe of $\mathcal{C}(G)$. In particular, $\mathcal{C}_c(G)$ is an algebraic frame.*

Proof. The main consequence of Lemma 3.1 is that since finite meets and arbitrary joins are the same in $\mathcal{C}_c(G)$ as in $\mathcal{C}(G)$, then $C_c(G)$ is a subframe of $\mathcal{C}(G)$. So this only leaves us with proving that $\mathcal{C}_c(G)$ is algebraic.

By definition, each c-subgroup is the directed join of principal convex ℓ-subgroups generated by complemented elements. Moreover, the principal convex ℓ-subgroups are the compact elements in $\mathcal{C}(G)$ and hence those in $\mathcal{C}_c(G)$ are compact in $\mathcal{C}_c(G)$. The interested reader can finish the proof that the compact elements of $\mathcal{C}_c(G)$ are precisely the principal convex ℓ-subgroups generated by complemented elements. Therefore, $\mathcal{C}_c(G)$ is an algebraic frame.

□

In Example 7.3, we provide a concrete example to show that the converse of (c) of Proposition 3.1 is not true. For now, we classify when the $g^{\perp}$ is a c-subgroup for all $g \in G$.

Proposition 3.3. *For all $g \in G$, $g^{\perp}$ is a c-subgroup if and only if G is weakly complemented.*

Proof. Suppose G is weakly complemented and let $0 \leq g \in G^+$. Let $h \in g^{\perp}$. By hypothesis, there is a complementary pair $x, y \in G^+$ such that $g \leq x$ and $h \leq y$. It follows that $g \wedge y = 0$, whence $y \in g^{\perp}$ is a complemented element for which $h \leq y$. Therefore, $g^{\perp}$ is a c-subgroup.

Conversely, suppose $g, h \in G$ satisfy $g \wedge h = 0$. Then $h \in g^{\perp}$, a c-subgroup, so there is a complemented element $y \in g^{\perp}$ with $h \leq y$. Let x be a complement of y and observe that so is $x' = x \vee g$. Thus, G is weakly complemented.

□

Remark 3.4. A consequence of Theorem 3.2 is that $\mathcal{C}(G) = \mathcal{C}_c(G)$ if and only if every principal convex ℓ-subgroup is a c-subgroup. But then either of these conditions occur if and only if G is complemented.

The embedding of $\mathcal{C}_c(G)$ into $\mathcal{C}(G)$ is a frame homomorphism and therefore, there is an adjoint map $h_* : \mathcal{C}(G) \longrightarrow \mathcal{C}_c(G)$. We expand on this. (For more information on the adjoint map of a frame homomorphism the reader may consult [17, Definition and Remarks 2.2].)

Starting with an $H \in \mathcal{C}(G)$ define

$$H_c = \{h \in H : |h| \leq e \text{ for some } e \in H^+ \cap c(G)\}.$$

Clearly, $H_c \subseteq H$. Furthermore, by Lemma 2.6, $H_c \in \mathcal{C}_c(G)$ and it is the largest c-subgroup contained in H. Another way of describing H_c is as the convex ℓ-subgroup of H generated by the complemented elements of H. It is possible that $H_c = \{0\}$ while $H \neq \{0\}$. By definition, H is a c-subgroup if and only if $H = H_c$. Moreover, the map $h_* : \mathcal{C}(G) \longrightarrow \mathcal{C}_c(G)$ is given by $h_*(H) = H_c$.

We are now ready to state our main result, characterizing when $\mathrm{Min}(G)^{-1}$ has a clopen π-base.

Theorem 3.5. *Suppose G is an ℓ-group. The following are equivalent.*

1. $\mathrm{Min}(G)^{-1}$ *has a clopen π-base.*
2. *For every non weak order unit $0 < g \in G$, there is a proper complemented element $e \in G^+$ such that $g \leq e$.*
3. *For every non weak order unit $0 < g \in G$, there is a non-zero complemented element f such that $f \wedge g = 0$.*
4. *For every non weak order unit $0 < g \in G$, $g_c^{\perp} \neq 0$.*

Proof. (1) and (2) are equivalent by Theorem 2.3.

(2) implies (3). Let $0 < g \in G$ be a non weak order unit. This means there is some $0 < h \in G$ such that $g \wedge h = 0$. By (2), there is some proper complemented $0 < e \in G$ such that $g \leq e$. Let $0 \leq f \in G$ be a complement of e. In fact, since e is proper $0 < f$. Then $e \wedge f = 0$ implies that $g \wedge f = 0$.

(3) implies (1). Let $V(g)$ be a basic open subset of $\mathrm{Min}(G)^{-1}$ with $0 \leq g$. We assume that $\emptyset \neq V(g) \subset \mathrm{Min}(G)$. It follows that g is not a weak order unit and $g \neq 0$. By (3), there is some complemented element $0 < f$ such that $f \wedge g = 0$. Note that f is not a weak order unit. Let $0 < e$ be a (proper) complement of f. A quick check ensures that $\emptyset \neq V(e) \subseteq V(g)$. Since e is complemented, $V(e)$ is a clopen subset of $\mathrm{Min}(G)^{-1}$. Consequently, $\mathrm{Min}(G)^{-1}$ has a clopen π-base.

The proof that (3) and (4) are equivalent is straightforward and left to the interested reader. □

Remark 3.6. Observe that the order of operation in the symbol $g_c^{\perp}$ (item (4)) is to first take the polar of g, and then take the largest c-subgroup inside of $g^{\perp}$. In general, this is not the same as taking the polar of the c-subgroup generated by $\mathfrak{G}(g)$. For example, if G has only trivial c-subgroups, then for a non weak order unit $(g^{\perp})_c = \{0\}$ whereas, $(\mathfrak{G}(g)_c)^{\perp} = G$.

4. The d-radical of an ℓ-group

It was an original thought on our part that Theorem 1.7 could be generalized by changing each instance of the phrase *is zero-dimensional* to *has a clopen π-base*. In trying to prove this we noticed that the intersection of all maximal d-subgroups is of importance. We continue to assume that G has weak order units. For the sake of ease, let $\mathfrak{w}(G)$ denote the set of positive weak order units of G.

Definition 4.1. Denote the intersection of all maximal d-subgroups by $\mathfrak{M}(G)$ and call this the **d-radical** of G.

Proposition 4.2. *Let G be an ℓ-group containing weak order units. Then*

$$\mathfrak{M}(G) = \{g \in G : \forall h \in G^+, h \vee |g| \in \mathfrak{w}(G) \text{ if and only if } h \in \mathfrak{w}(G)\}.$$

Proof. Suppose $g \in G$ has the property that if $h \vee |g| \in \mathfrak{w}(G)$, then $h \in \mathfrak{w}(G)$. If $g \notin \mathfrak{M}(G)$, then there is some maximal d-subgroup $M \in \mathrm{Max}_d(G)$ such that $g \notin M$. The join of $\mathfrak{G}(g)$ and M must therefore contain a weak order unit. It follows from the Riesz Representation Theorem and the Triangle Inequality, that for some finite set $0 \leq m_1, \ldots, m_n \in M$ and $k_1 \ldots, k_n \in \mathbb{N}$, the element

$$k_1 g + m_1 + k_2 + m_2 + \ldots + k_n g + m_n \in \mathfrak{w}(G).$$

Subsequently, there is some $0 \leq g' \in \mathfrak{G}(g)$ and $0 \leq m \in M$ such that $g' \vee m$ is a weak order unit. But then so is $|g| \vee m$. So, by choice of g, $m \in M$ is a weak order unit, a contradiction. Therefore, $g \in \mathfrak{M}(G)$.

Conversely, let $g \in \mathfrak{M}(G)$. Obviously, if $h \in \mathfrak{w}(G)$, then so is $h \vee |g| \in \mathfrak{w}(G)$. So suppose that $h \in G^+$ and $h \vee g \in \mathfrak{w}(G)$. If h is not a weak order unit, then there is some $M \in \mathrm{Max}_d(G)$ such that $h \in M$. But then so is $h \vee |g|$, a contradiction. Therefore, $h \in \mathfrak{w}(G)$. □

A natural question is whether $\mathfrak{M}(G) \trianglelefteq G$. We demonstrate this now. The first two results of the next proposition can be found in [6].

Proposition 4.3. *Let G be an ℓ-group.*

(a) *Conjugation is an ℓ-isomorphism.*
(b) *For each $x, g \in G$, $x + g^{\perp\perp} - x = (x + g - x)^{\perp\perp}$.*
(c) *For each $x \in G$, if H is a d-subgroup, then $x + H - x$ is a d-subgroup.*
(d) *For each $x \in G$ and $M \in \mathrm{Max}_d(G)$, $x + M - x \in \mathrm{Max}_d(G)$.*

Proof. (c) Suppose $x \in G$ and let $y \in x + H - x$. Then $-x + y + x \in H$, whence

$$-x + y^{\perp\perp} + x = (-x + y + x)^{\perp\perp} \subseteq H.$$

Therefore, $y^{\perp\perp} \subseteq x + H - x$.

(d) Let $M \in \mathrm{Max}_d(G)$ and take any d-subgroup H containing $x + M - x$. Then $M \subseteq -x + H + x$, the latter being a d-subgroup. It follows that $M = -x + H + x$, whence $x + M - x = H$. So, $x + M - x \in \mathrm{Max}_d(G)$. □

The following is a consequence of Proposition 4.3 (d).

Proposition 4.4. *Let G be an ℓ-group containing a weak order unit. Then the d-radical of G is a normal subgroup of G, i.e. $\mathfrak{M}(G) \trianglelefteq G$.*

Definition 4.5. For the lack of a better term we call an element $0 < g \in G$ **left fusible** if it can be written as the sum of a weak order unit and a non weak order unit. An ℓ-group is called **left fusible** if it has the property that every nonzero positive element is left fusible. We define a right fusible element and ℓ-group analogously. A positive element that is both left and right fusible will be called fusible. It ought to be clear what we mean by a fusible ℓ-group.

Clearly, (positive) weak order units are fusible and 0 is never considered fusible.

Proposition 4.6. *Let G be an ℓ-group containing a weak order unit. The following statements are equivalent.*

(1) *The d-radical of G is zero.*
(2) *For each $0 < g \in G$ there is some non weak order unit $h \in G^+$ such that $g \vee h$ is a weak order unit.*
(3) *G is left fusible.*
(4) *G is fusible.*
(5) *G is right fusible.*

Proof. (1) implies (2). Let $0 < g \in G$. If g is a weak order unit, then $h = 0$ works. Otherwise, by hypothesis, $g \notin \mathfrak{M}(G)$. Applying Proposition 4.2 and observing that $h \in \mathfrak{w}(G)$ always implies $g \vee w \in \mathfrak{w}(G)$, then this means there is some non weak order unit $0 < h \in G$ such that $g \vee h$ is a weak order unit.

(2) implies (3). Let $0 < g \in G^+$ be a non weak order unit. By (2), there is some non weak order unit $h \in G^+$ such that $h \vee g \in \mathfrak{w}(G)$. Since

$$\mathfrak{G}(g+h) = \mathfrak{G}(g \vee h) = \mathfrak{G}(h) \vee \mathfrak{G}(g)$$

it follows that $g + h \in \mathfrak{w}(G)$. Therefore, $g = (g+h) + (-h)$ is a left fusible representation of g.

(3) implies (1). Let $0 < g \in G$. We aim to show that $g \notin \mathfrak{M}(G)$, which by Proposition 4.2 is tantamount to finding a non weak order unit $h \in G^+$ such that $g \vee h \in \mathfrak{w}(G)$. By (3), there is some weak order unit $w \in G$ and non weak order unit $m \in G$ such that $g = w + m$. Now,

$$w = g - m \in \mathfrak{G}(g) \vee \mathfrak{G}(m) = \mathfrak{G}(g) \vee \mathfrak{G}(|m|) = \mathfrak{G}(g \vee |m|).$$

It follows that $g \vee |m|$ is a weak order unit. Since m is not a weak order unit, neither is $|m|$. Consequently, $g \notin \mathfrak{M}(G)$ and so $\mathfrak{M}(G) = \{0\}$.

The rest of the proof follows in the same vein once you observe that the join operation is commutative.

□

Remark 4.7. Notice that every complemented ℓ-group is fusible and thus, so is every projectable ℓ-group. Those familiar with the analogy of ℓ-groups to semi-prime commutative rings with identity can attest to the fact that this situation is analogous to saying that every non-zero element can be written as the sum of a (left) zero-divisor and a non-zero-divisor. Such rings are called *left fusible* (see [9]). The slight difference is that in a ring addition is commutative, though multiplication need not be. There are left fusible rings that are not right fusible.

In most of the previous work done on maximal d-subgroups authors have been interested in archimedean ℓ-groups. In the archimedean case, it can be shown that $\mathfrak{M}(G) = \{0\}$. Some time ago, the question arose of whether an archimedean ℓ-group with weak order unit is fusible. Professor A.W. Hager provided a proof which directly showed that such an object is fusible. Here we supply a different proof.

Proposition 4.8. *Suppose G is an archimedean ℓ-group with a weak order unit. Then $\cap \operatorname{Max}_d(G) = \{0\}$, and such an ℓ-group is fusible.*

Proof. Let G be an ℓ-group and $0 < u \in G$ a weak order unit. We begin by demonstrating that any element in $\cap \operatorname{Max}_d(G)$, say g, also belongs to $\cap \operatorname{Val}(u)$. To that end, let $V \in \operatorname{Val}(u)$ and choose a minimal prime beneath V, say P. Since P does not contain any weak order units it follows that P can be extended to a convex ℓ-subgroup M which is maximal with respect to not containing any weak order units, i.e. a maximal d-subgroup ([5, Proposition 4.3]), i.e. $M \in \operatorname{Max}_d(G)$. Since $u \notin M$, then we can extend M to a value of u, which must be V since $\operatorname{Spec}(G)$ is a root system. Therefore, $M \subseteq V$. Since $g \in M$, then $g \in V$. Therefore, $\cap \operatorname{Max}_d(G) \subseteq \cap \operatorname{Val}(u)$.

Finally, archimedean ℓ-groups have the property that $\cap \operatorname{Val}(u) = \{0\}$. Therefore, an archimedean ℓ-group is fusible. □

We end this section with a now obvious characterization of the d-radical of an ℓ-group.

Proposition 4.9. *Let G be an ℓ-group. Then*

$$\mathfrak{M}(G) = \{g \in G : |g| \textit{ is not fusible } \}.$$

5. When $\mathrm{Max}_d(G)$ has a clopen π-base

Interestingly, to classify when $\mathrm{Max}_d(G)$ has a clopen π-base one needs a result similar to Lemma 2.2. Unfortunately, the existence of non-zero non-fusible elements in an arbitrary ℓ-group muddles up the situation.

Lemma 5.1. *Let G be an ℓ-group with a weak order unit. Let $K \subseteq \mathrm{Max}_d(G)$. Then K is clopen if and only if there is a pair $g, h \in G^+$ such that $g \wedge h \in \mathfrak{M}(G)$ and $g \vee h$ is a weak order unit and $K = U_d(g)$. In particular, if $e \in c(G)$, then $U_d(e)$ is a clopen subset of $\mathrm{Max}_d(G)$.*

Furthermore, if $K = U_d(g)$ is clopen for $g \in G^+$, then there is some $h \in G^+$ such that $g \wedge h \in \mathfrak{M}(G)$ and $g \vee h \in \mathfrak{w}(G)$.

Lastly, if G is fusible, then for any $0 \leq g$, $U_d(g)$ is clopen if and only if g is complemented.

Proof. The last two statements certainly follow from the first paragraph.

Let K be a clopen subset of $\mathrm{Max}_d(G)$. Recall that $\mathrm{Max}_d(G)$ is compact Hausdorff so that $K = U_d(g)$ for some $g \in G^+$. Similarly, $\mathrm{Max}_d(G) \smallsetminus K = U_d(h)$ for some $h \in G^+$. It follows that $U_d(g \vee h) = U_d(g) \cup U_d(h) = \mathrm{Max}_d(G)$ so that $g \vee h \in \mathfrak{w}(G)$. Now, $U_d(g \wedge h) = \emptyset$ implying that $g \wedge h \in \mathfrak{M}(G)$.

Conversely, suppose g, h satisfy the condition that $g \wedge h \in \mathfrak{M}(G)$ and $g \vee h$ is a weak order unit and $K = U_d(g)$. Then,

$$U_d(g) \cap U_d(h) = U_d(g \wedge h) = \emptyset$$

and

$$U_d(g) \cup U_d(h) = U_d(g \vee h) = \mathrm{Max}_d(G).$$

Consequently, $K = U_d(g)$ is a clopen subset of $\mathrm{Max}_d(G)$.

Finally, suppose $K = U_d(g)$ is clopen. Then for any $h \in G^+$ for which $U_d(h) = \mathrm{Max}_d(G) \smallsetminus U_d(h)$ will also satisfy that $g \wedge h \in \mathfrak{M}(G)$ and $g \vee h \in \mathfrak{w}(G)$. ☐

In order to characterize when $\mathrm{Max}_d(G)$ has a clopen π-base we first consider when G is fusible. Notice the dual nature of our next proposition in comparison to Theorem 2.3.

Proposition 5.2. *The ℓ-group G is fusible and the space $\mathrm{Max}_d(G)$ has a clopen π-base if and only if for each $0 < g \in G$, there exists an $e \in c(G)$ such that $0 < e \leq g$.*

Proof. First, suppose that $\mathrm{Max}_d(G)$ has a clopen π-base and G is fusible. Let $0 < g \in G$. If g is a weak order unit, then we are done. Otherwise, by Lemma 5.1, choose a complemented element $0 < e \in G$ such that $\emptyset \neq U_d(e) \subseteq U_d(g)$. Then

$U_d(e) = U_d(e) \cap U_d(g) = U_d(e \wedge g)$, so $e \wedge g$ is complemented as G is fusible. Since $U_d(e)$ is not empty it follows that $0 < e \wedge g \leq g$.

Conversely, suppose that for each $0 < g \in G$, there exists an $e \in c(G)$ such that $0 < e \leq g$. Let $U_d(g)$ be a basic non-empty subset of $\mathrm{Max}_d(G)$. Without loss of generality, $0 < g$. By hypothesis there is some $0 < e \in c(G)$ such that $0 < e \leq g$. Then $U_d(e)$ is a non-empty clopen subset of $\mathrm{Max}_d(G)$ and

$$U_d(e) = U_d(e \wedge g) = U_d(e) \cap U_d(g) \subseteq U_d(g).$$

Consequently, $\mathrm{Max}_d(G)$ has a clopen π-base.

Next, let $0 \leq g \in \mathfrak{M}(G)$. If $0 < g$ then we can choose $e \in c(G)$ such that $0 < e \leq g$. Since $\mathfrak{M}(G)$ is a convex ℓ-subgroup it follows that $e \in \mathfrak{M}(G)$, contradicting that e is complemented. Therefore, $\mathfrak{M}(G) = \{0\}$, i.e. G is fusible. □

Remark 5.3. Recall that if G is lamron, then $\mathrm{Min}(G)^{-1}$ and $\mathrm{Max}_d(G)$ are homeomorphic. Therefore, one of them has a clopen π-base precisely when the other does. Since we cannot determine whether a lamron ℓ-group is fusible, we are not satisfied with our next theorem. On the bright side, the case does cover a lot of ground.

Remark 5.4. Recall that in Remark 3.4, the map $h_* : \mathcal{C}(G) \longrightarrow \mathcal{C}_c(G)$ was defined by $h_*(H) = H_c$. The authors in [17] defined such a map h_* to be **$*$-dense** if $h_*(H) = 0$ implies $H = 0$. We remarked in Section 3 that it is possible that $H_c = \{0\}$ while $H \neq \{0\}$, that is, it is possible for the embedding $\mathcal{C}_c(G) \longrightarrow \mathcal{C}(G)$ not to be $*$-dense. In the result that follows we show, amongst other things, that the $*$-density of this frame homomorphism is equivalent to $\mathrm{Min}(G)^{-1}$ having a clopen π-base.

Theorem 5.5. *Let G be a fusible ℓ-group. The following statements are equivalent.*

1. $\mathrm{Min}(G)^{-1}$ *has a clopen π-base.*
2. *For each $0 < g \in G$, there exists a proper complemented element $e \in c(G)$ such that $0 < g \leq e$.*
3. $\mathrm{Max}_d(G)$ *has a clopen π-base.*
4. *For each $0 < g \in G$, there exists an $e \in c(G)$ such that $0 < e \leq g$.*
5. *The embedding of $\mathcal{C}_c(G)$ into $\mathcal{C}(G)$ is $*$-dense.*

In particular, if G is an archimedean ℓ-group, then $\mathrm{Min}(G)^{-1}$ *has a clopen π-base if and only if* $\mathrm{Max}_d(G)$ *has a clopen π-base.*

Proof. (1) and (2) are equivalent for all ℓ-groups (Theorem 2.3). It is also straightforward to check that (4) and (5) are equivalent for all ℓ-groups. Theorem 5.2 says that (3) and (4) are equivalent for all fusible ℓ-groups.

Recall from our discussion prior to Proposition 1.7, that the map $\mathfrak{d} : \mathrm{Min}(G)^{-1} \longrightarrow \mathrm{Max}_d(G)$ is a continuous surjection.

(1) implies (3). Let $U_d(g)$ be a basic non-empty open subset of $\mathrm{Max}_d(G)$ with $0 < g \in G$. Choose $M \in U_d(g)$ and let $P \in \mathrm{Min}(G)$ for which $\mathfrak{d}(P) = M$. Notice that $P \in \mathfrak{d}^{-1}(U_d(g))$, the latter being an open subset of $\mathrm{Min}(G)^{-1}$ by continuity of $\mathfrak{d}$. Thus, there is some $0 < h \in G$ so that $P \in V(h) \subseteq \mathfrak{d}^{-1}(U_d(g))$. By hypothesis, there is a non-empty clopen subset of $\mathrm{Min}(G)^{-1}$, say $U(e)$, such that $\emptyset \neq U(e) \subseteq V(h)$.

Since e is complemented it follows that $U_d(e) = \mathfrak{d}(U(e))$, which is a non-empty clopen subset of $\mathrm{Max}_d(G)$. Furthermore, $U_d(e) \subseteq U_d(g)$. Consequently, $\mathrm{Max}_d(G)$ has a clopen π-base

(4) implies (2). Suppose for each $0 < g \in G$, there exists an $e \in c(G)$ such that $0 < e \leq g$. Let $0 < g \in G$ and assume without loss of generality that g is not a weak order unit, and so there exists $0 < h \in g^{\perp}$. By hypothesis, there exists a complemented element $0 < e \leq h$. Let f be a complement of e so that $f \wedge e = 0$ and $f \vee e$ is a weak order unit. Set $f' = g \vee f$; clearly $g \leq f'$. Now, $f \vee e \leq f' \vee e$, whence $f' \vee e$ is also a weak order unit. Also,

$$f' \wedge e = (g \vee f) \wedge e = (g \wedge e) \vee (f \wedge e) = 0.$$

It follows that f' is a proper complemented element above g. □

Remark 5.6. We observed above that a continuous surjection of a topological space with a clopen π-base has a clopen π-base. Therefore, the content of the above proof corroborates that the topologies of $\mathrm{Min}(G)^{-1}$ and $\mathrm{Max}_d(G)$ are closely aligned.

Remark 5.7. A thorough inspection of Theorem 5.5, reveals that condition (4) is the strongest. Condition (4) implies that $\mathrm{Min}(G)^{-1}$ has a clopen π-base, which in turn implies that $\mathrm{Max}_d(G)$ has a clopen π-base. There are examples of ℓ-groups G for which $\mathrm{Max}_d(G)$ has a clopen π-base yet $\mathrm{Min}(G)^{-1}$ does not. Also, imposing the lamron condition yields that conditions (1), (2), and (3) are equivalent, and we do not know whether they in turn imply (4).

Question 5.8. It ought to be apparent that a weakly complemented ℓ-group is fusible. We have been unable to show that a lamron ℓ-group is fusible, even for abelian ℓ-groups. We also do not know whether an ℓ-group with stranded primes is fusible. We guess not in both cases.

6. W

The work in this section takes place in the category **W**. Objects in **W** are pairs (G, u) where G is an archimedean lattice-ordered group and u is a distinguished (positive) weak order unit. A morphism in **W**, between (G, u) and (H, v), is an ℓ-group homomorphism $\varphi : G \longrightarrow H$ such that $\varphi(u) = v$. One of the main features of this category is seen through the Yosida Embedding Theorem.

Recall that the Yosida space of a **W**-object (G, u) is the space of values of u; this set is denoted by YG. This space is always compact and Hausdorff when equipped with the hull-kernel topology. A basic open set is of the form

$$\mathrm{coz}(g) = \{p \in YG : g \notin P\}$$

for some $g \in G$. The set $\mathrm{coz}(g)$ is called the **cozero-set of** g and the collection of all such cozero-sets is termed the set of G-cozero-sets and is denoted by $\mathrm{coz}(G)$. The complement of $\mathrm{coz}(g)$ is denoted by $Z(g)$ and is called the **zero-set** of g. The collection of G-zero-sets is denoted by $Z(G)$. It ought to be clear that $\mathrm{coz}(g \vee f) = \mathrm{coz}(g) \cup \mathrm{coz}(f)$ and $\mathrm{coz}(g \wedge f) = \mathrm{coz}(g) \cap \mathrm{coz}(f)$ for all $f, g \in G^+$. Thus, both $\mathrm{coz}(G)$ and $Z(G)$ are lattices under inclusion.

Definition 6.1. Recall that for a compact Hausdorff space X, the set $D(X)$ denotes the collection of all **almost real-valued** continuous functions on X. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\}$ denote the two point compactification of the space of reals. An element $f \in D(X)$ has the feature that $f : X \longrightarrow \overline{\mathbb{R}}$ is continuous and $f^{-1}(\mathbb{R})$ is a dense subset of X. This set need not be a group as addition might not make sense, but it is always a lattice. To say that H is an ℓ-subgroup of $D(X)$ implies that H is in fact closed under addition.

Corollary 6.2. *Let (G, u) be a $\mathbf{W}$-object. For all $0 < v \in G$, v is a weak order unit of G if and only if $\operatorname{coz}(v)$ is a dense subset of YG.*

Proof. The proof of this is well-known, but we include it here for completeness sake.

Let $0 < v \in G$ be a weak order unit. Let $f \in G$ satisfy $\operatorname{coz}(f) \cap \operatorname{coz}(v) = \emptyset$. Then $\operatorname{coz}(f \wedge v) = \emptyset$, whence $f \wedge v = 0$ so that $f = 0$. Therefore, $\operatorname{coz}(v)$ is a dense subset of YG.

Conversely, if $\operatorname{coz}(v)$ is a dense subset, then for any $0 < f \in G$, $\operatorname{coz}(v \wedge f) = \operatorname{coz}(f) \cap \operatorname{coz}(v) \neq \emptyset$. But then $v \wedge f \neq 0$, so that v is a weak order unit. □

Theorem 6.3 (The Yosida Embedding Theorem)**.** *Let (G, u) be a $\mathbf{W}$-object. There is an ℓ-isomorphism of G ($g \mapsto \hat{g}$) onto an ℓ-subgroup $\hat{G} \leq D(YG)$ such that $\hat{u} = \mathbf{1}$ and $\hat{G}$ has the following separation property: for each $p \in YG$ and closed set $V \subseteq YG$ not containing p, there is some $g \in G$ for which $\hat{g}(p) = 1$ and $\hat{g}(q) = 0$ for all $q \in V$. Moreover, YG is the unique compact space, up to homeomorphism, satisfying these two properties.*

Example 6.4. The prototypical example of a $\mathbf{W}$-object is $C(X)$, the set of continuous real-valued functions on a topological space X. We assume that X is Tychonoff, that is, completely regular and Hausdorff. It shall be assumed, unless otherwise noted, that when considering $C(X)$ as a $\mathbf{W}$-object, that the constant function $\mathbf{1}$ is the distinguished weak order unit. In this case, the Yosida space of $(C(X), \mathbf{1})$ is the Stone-Čech compactification of X, βX.

For $f \in C(X)$, the **cozero-set** of f is $\operatorname{coz}(f) = \{x \in X : f(x) \neq 0\}$. A cozero-set of the space X is a set of the form $\operatorname{coz}(f)$ for some $f \in C(X)$. The collection of all cozero-sets (resp. zero-sets) of X is denoted by $\operatorname{coz}(X)$ (resp. $Z(X)$). Notice that a cozero-set of X is not necessarily a $C(X)$-cozero-set of f as the latter is a subset of βX. We do observe that when X is compact then the notions coincide. Moreover, for any $\mathbf{W}$-object (G, u), each G-cozero-set (resp. G-zero-set) is a cozero-set (resp. zero-set) of YG. For a more thorough explanation of this see [4, Example 3.2].

Definition 6.5. Let $(G, u) \in \mathbf{W}$. The following collection of regular closed subsets of YG play a pivotal role in the classification of classes of $\mathbf{W}$-objects. The interested reader can also check [5].

(1) $\mathcal{R}(YG) = \{V \subseteq YG : V = \operatorname{cl}\operatorname{int} V\}$.
(2) $Z^\sharp(G) = \{\operatorname{cl}\operatorname{int} Z(g) : f \in G^+\}$.
(3) $\operatorname{cl}\operatorname{coz}(G) = \{\operatorname{cl}\operatorname{coz}(g) : g \in G^+\}$.
(4) $cc(G) = \{\operatorname{coz}(e) : e \in c(G)\}$.
(5) $\mathscr{G}(G) = \{\operatorname{cl} C : C \in cc(G)\}$.

(6) $\mathrm{Clop}(G) = \{K \subseteq YG : K \text{ is a clopen subset of } YG\} = \mathrm{Clop}(YG)$.

When X is a compact space and $G = C(X)$, then instead we write $Z^\sharp(X)$, $\mathrm{cl}\,\mathrm{coz}(X)$, $cc(X)$, and $\mathscr{G}(X)$.

For any **W**-object (G, u) we know that $Z^\sharp(G) \subseteq Z^\sharp(YG)$, $\mathrm{Clop}(G) \subseteq cc(G) \subseteq \mathrm{cl}\,\mathrm{coz}(G) \subseteq \mathrm{cl}\,\mathrm{coz}(YG)$, and $\mathscr{G}(G) \subseteq cc(YG)$. When ordered by inclusion $\mathcal{R}(YG)$ is a complete boolean algebra. The lattice operations are given as follows.

(i) $V_1 \cup' V_2 = V_1 \cup V_2$;
(ii) $V_1 \cap' V_2 = \mathrm{cl}\,\mathrm{int}(V_1 \cap V_2)$;
(iii) $V' = \mathrm{cl}(YG \smallsetminus V)$.

Observe that the above lattice operations make $\mathscr{G}(G)$ into a boolean algebra. Furthermore, the equality in item (iii) yields that the set of complements of $Z^\sharp(G)$ in $\mathcal{R}(YG)$ is precisely $\mathrm{cl}\,\mathrm{coz}(G)$. It follows that either of $Z^\sharp(G)$ or $\mathrm{cl}\,\mathrm{coz}(G)$ is a boolean algebra if and only if $Z^\sharp(G) = \mathrm{cl}\,\mathrm{coz}(G)$.

The following three results are very useful in doing calculations on the just-defined objects. The results are stated in terms of a compact Hausdorff space and therefore hold for the Yosida space of any **W**-object.

Lemma 6.6. *Let X be a compact Hausdorff space and let $Z, Z_1, Z_2 \in Z(X)$. The following hold.*

(a) $\mathrm{cl}\,\mathrm{int}\, Z_1 \cap' \mathrm{cl}\,\mathrm{int}\, Z_2 = \mathrm{cl}\,\mathrm{int}(Z_1 \cap Z_2)$.
(b) $\mathrm{cl}\,\mathrm{int}\, Z_1 \cup' \mathrm{cl}\,\mathrm{int}\, Z_2 = \mathrm{cl}\,\mathrm{int}\, Z_1 \cup \mathrm{cl}\,\mathrm{int}\, Z_2 = \mathrm{cl}\,\mathrm{int}(Z_1 \cup Z_2)$.
(c) $(\mathrm{cl}\,\mathrm{int}\, Z)' = \mathrm{cl}(X \smallsetminus Z)$.

Lemma 6.7. *Let X be a compact space and let $f, g \in C(X)^+$. The following hold.*

(a) $\mathrm{cl}\,\mathrm{coz}(f) \cap' \mathrm{cl}\,\mathrm{coz}(g) = \mathrm{cl}(\mathrm{coz}(f) \cap \mathrm{coz}(g)) = \mathrm{cl}\,\mathrm{coz}(f \wedge g)$.
(b) $\mathrm{cl}\,\mathrm{coz}(f) \cup' \mathrm{cl}\,\mathrm{coz}(g) = \mathrm{cl}(\mathrm{coz}(f) \cup \mathrm{coz}(g)) = \mathrm{cl}\,\mathrm{coz}(f \vee g)$.

Corollary 6.8. *Let (G, u) be a **W**-object. Then each of $Z^\sharp(G), \mathrm{cl}\,\mathrm{coz}(G)$, and $cc(G)$ is a sub-lattice of $\mathcal{R}(YG)$.*

Definition 6.9. Recall that given a boolean algebra $\mathscr{A}$, a sub-boolean algebra, $\mathscr{B}$, is said to be **dense** in $\mathscr{A}$, if for every non-zero $0 < a \in A$, there is a nonzero $0 < b \in B$ such that $b \leq a$.

The density property has been used to characterize completions of boolean algebras. In particular, $\mathscr{A}$ is a completion of $\mathscr{B}$ if $\mathscr{A}$ is a complete boolean algebra and $\mathscr{B}$ is dense in A.

We are now in position to provide some different ways of looking at the situation of when $\mathrm{Max}_d(G)$ has a clopen π-base. What is new to this theorem (see Theorem 5.5) is the connection to the Yosida space of (G, u).

Theorem 6.10. *Let (G, u) be a **W**-object. The following statements are equivalent.*

(1) $\mathrm{Max}_d(G)$ *has a clopen π-base.*
(2) $\mathrm{Min}(G)^{-1}$ *has a clopen π-base.*
(3) *For each $0 < g \in G$, there exists an $e \in c(G)$ such that $0 < e \leq g$.*
(4) *For each $0 < g \in G$, there exists a proper $e \in c(G)$ such that $0 < g \leq e$.*
(5) *The collection $cc(G)$ is an open π-base of YG.*

(6) *The sub-boolean algebra* $\mathscr{G}(G)$ *is dense in* $\mathcal{R}(YG)$.
(7) *The collection of interiors of complemented* G*-zero-sets is an open* π*-base of* YG.

Proof. Clearly, (1), (2), (3), and (4) are equivalent since a **W**-object is fusible: Theorem 5.5.

(1) implies (5). Let O be a non-empty open subset of YG. Choose $0 < g \in G^+$ such that $\emptyset \neq \mathrm{coz}(g) \subseteq O$. By (3), there is a complemented $0 < e \leq g$. Observe that $\emptyset \neq \mathrm{coz}(e) \subseteq \mathrm{coz}(g)$. Consequently, $cc(G)$ is a π-base for YG.

(5) implies (6). Let $\emptyset \neq V \in \mathcal{R}(YG)$, a non-empty regular closed subset. By (5) we can choose $0 < e \in c(G)$ such that $\mathrm{coz}(e) \subseteq \mathrm{int}\, V$. Then $\emptyset \neq \mathrm{cl}\, \mathrm{coz}(e) \subseteq \mathrm{cl}\, \mathrm{int}\, V = V$. Therefore, $\mathscr{G}(G)$ is dense in $\mathcal{R}(YG)$.

(6) implies (7). Let $O \subseteq YG$ be a non-empty open subset. We can shrink O down to a non-empty G-cozero-set, say $\mathrm{coz}(g)$, such that $\mathrm{cl}\, \mathrm{coz}(g) \subseteq O$. Since $\mathrm{cl}\, \mathrm{coz}(g) \in \mathcal{R}(YG)$, we can apply (6) and choose some $e \in C(G)$ such that $\emptyset \neq \mathrm{cl}\, \mathrm{coz}(e) \subseteq \mathrm{cl}\, \mathrm{coz}(g)$. Let $f \in c(G)$ be a complement of e. It is straightforward to check that $\emptyset \neq \mathrm{int}\, Z(f) \subseteq \mathrm{cl}\, \mathrm{coz}(e)$, whence $\mathrm{int}\, Z(f) \subseteq O$.

(7) implies (3). Let $0 < g' \in G$. As mentioned before, we can shrink down $\mathrm{coz}(g')$ to a G-cozeroset, say $\mathrm{coz}(g)$, so that

$$\mathrm{coz}(g) \subseteq \mathrm{cl}\, \mathrm{coz}(g) \subseteq \mathrm{coz}(g').$$

By (7), there is a complemented element $f \in c(G)$ such that $\emptyset \neq \mathrm{int}\, Z(f) \subseteq \mathrm{coz}(g)$. Let $e \in c(G)$ be a complement of f and note that $\mathrm{coz}(e) \subseteq \mathrm{int}\, Z(f)$. It follows that $\mathrm{coz}(e) = \mathrm{coz}(e) \cap \mathrm{coz}(g) = \mathrm{coz}(e \wedge g)$. Thus, $e' = e \wedge g$ has the property that $e' \wedge f = 0$ and $e' \vee f$ is a weak order unit (Corollary 6.2). Therefore, e' is a complemented element and $0 < e' \leq g$.

□

7. Examples

We recall some of the examples from [5] and supply some new ones to help round out the theory.

Example 7.1. In the paragraph after Lemma 2.6, it was mentioned that there are examples of ℓ-groups whose non-zero complemented elements are precisely the weak order units. These are precisely the ℓ-groups so that $\mathrm{Min}(G)^{-1}$ is connected. If the ℓ-group G is fusible, then $\mathrm{Min}(G)^{-1}$ is connected if and only if $\mathrm{Max}_d(G)$ is connected.

In the context of **W**, the above is characterized by the property on YG that says YG is connected and there are no proper dense G-cozero-sets; the latter half of this is covered in [5, Theorem 5.3]. This happens if and only if $\mathrm{Max}_d(G) = YG$. For $C(X)$ this is saying that βX is a connected almost P-space, which is equivalent to saying that X is a connected pseudo-compact almost P-space ([15, Proposition 2.2]). (A space X is an **almost** P**-space** if it has no proper dense cozero-sets.) The space $Z = \beta[0,1] \smallsetminus [0,1]$ is a connected compact almost P-space.

Example 7.2. Recall Example 1.6, $G = \overrightarrow{\mathbb{Z} \times \oplus\mathbb{Z}}$ is the lexicographical extension of $\mathbb{Z}$ over H, the direct sum of countable many copies of $\mathbb{Z}$. This is not a fusible ℓ-group since H is the maximal d-subgroup. Moreover, in this case, $\mathrm{Min}(G)^{-1}$ is homeomorphic to the naturals equipped with the co-finite topology which does not have a clopen π-base. However, $\mathrm{Max}_d(G)$ does have a clopen π-base, trivially.

This construction can be generalized to any H with no weak order unit and obtain that, the ℓ-group $G = \overrightarrow{\mathbb{Z} \times H}$ satisfies that the space $\mathrm{Min}(G)^{-1}$ is connected. As mentioned before G is not fusible, and G has a unique maximal d-subgroup; $\mathrm{Max}_d(G)$ $=\{H\}$ is trivially connected. Observe that if H has a weak order unit, then H is not a maximal d-subgroup of G.

If G has a unique maximal d-subgroup, say K, then G is a lex extension of K and so $\mathrm{Min}(K)^{-1}$ is homeomorphic to $\mathrm{Min}(G)^{-1}$. Moreover, K must have not any weak order units and so this is the case as above.

Example 7.3. The converse to (c) of Proposition 3.1 is not true. Namely, that it is possible that $f^{\perp}$ is a c-subgroup without f being a complemented element.

Let X be the space obtained by taking $\alpha\mathbb{N}$ and ω_1^* (the space of countable ordinals together with ω_1) and gluing at the points α and ω_1. The function f which maps the natural $n \in \alpha\mathbb{N}$ to $\frac{1}{n}$ and everything in ω_1^* to 0 is not a complemented element. However, any function $0 < g \in f^{\perp}$ must send ω_1 to 0 and therefore be 0 on an interval around ω_1. Therefore, the cozero-set of g is contained in a proper clopen subset of ω_1^* and so g is beneath some multiple of a characteristic function belonging to $f^{\perp}$; such an element happens to be a complemented element. Consequently, $f^{\perp}$ is a c-subgroup. One can check that there is no complemented element g such that $f^{\perp} = g^{\perp}$.

Question. Reading Remark 5.8 once again, we are left with the question of whether for a general ℓ-group G condition (4) of Theorem 5.5 is equivalent to $\mathrm{Min}(G)^{-1}$ possessing a clopen π-base. Notice that condition (4) is equivalent to the statement that $G_c \leq G$ is a dense extension, which is sufficient for G to be fusible. Therefore, we are left with the question of whether there is a non-fusible ℓ-group with $\mathrm{Min}(G)^{-1}$ having a clopen π-base. If T is a totally ordered group and H is an ℓ-group, then $\mathrm{Min}(\overrightarrow{T \times H})^{-1}$ is homeomorphic to $\mathrm{Min}(H)^{-1}$. Thus, the use of lexicographical extensions seems to not be useful in constructing such a group, Since laterally complete ℓ-groups are complemented, this also rules out the typical constructions like $\mathrm{Aut}(\Omega)$ and Hahn groups.

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